

Homological dimensions and strongly idempotent ideals

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Abstract

Let A be an Artin algebra and e an idempotent in A . It is an interesting topic to compare the homological dimension of the algebras $A, A/AeA$ and eAe . For example, in [2], the relation among the global dimension of these algebras is discussed under the condition that AeA is a strongly idempotent ideal. Motivated by this, we try to compare the finitistic dimension of these algebras under certain homological conditions on AeA . In particular, under the condition that AeA is a strongly idempotent ideal with finite projective dimension, we prove that if the finitistic projective (or injective) dimension of eAe and A/AeA are finite, then the finitistic projective (or injective) dimension of A is finite. This is a generalized version of the main result in [1].

1 Introduction

The open finitistic dimension conjecture says that the finitistic dimension of any Artin algebra is finite (see [3]). It is known that a positive answer to this conjecture will imply the solutions to other homological conjectures (see [10]). Thus, it is one of the main topics in the representation theory of Artin algebras to study the finiteness of the finitistic dimension of Artin algebras and it has been proved that several classes of algebras have finite finitistic dimension. For details, we refer to [9, 11] and the references therein. Let A be an Artin algebra and e an idempotent in A . In [1], it is proved that if e is primitive idempotent in A such that AeA is projective, then the finiteness of the finitistic dimension of A/AeA implies that of the finitistic dimension of A . Further, the result implies that the finitistic dimension of standardly stratified algebras is finite. On the other hand, it is also an interesting topic to compare the homological dimension of the three algebras $A, A/AeA$ and eAe . In general, it is difficult for us to do it directly. For this reason, the work usually has been done under certain homological conditions on the ideal AeA (see [4, 5, 6]). For example, the global dimension of these algebras are compared under the condition that AeA is a strongly idempotent ideal in [2].

Motivated by the statement above, in the paper, we try to study the relation among the finiteness of the finitistic dimension of $A, A/AeA$ and eAe by comparing the finitistic dimension of these algebras. We shall mainly do this under the condition that AeA is a strongly idempotent ideal with finite projective dimension. Recall that AeA is called a strongly idempotent ideal if the epimorphism $A \rightarrow A/AeA$ induces isomorphisms

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$\rho_{X,Y}^n : \text{Ext}_{A/AeA}^n(X, Y) \rightarrow \text{Ext}_A^n(X, Y)$ for all $n > 0$ and for all A/AeA -modules X and Y . A simple example of strongly idempotent ideals is the ideal AeA which is projective on either side. Moreover, if we just need that e is an idempotent which is not necessarily primitive, then we can construct examples of strongly idempotent ideal which are not projective but of finite projective dimension. For example, let $0 = I_0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = A$ be a chain of idempotent ideals of A such that I_{k+1}/I_k is a projective A/I_k -module for $0 \leq k \leq n-1$. Then I_k is a strongly idempotent ideal with finite projective dimension for $1 \leq k \leq n$ (see [2]). This implies that our consideration is completely new and more general than that in [1]. Denote $\text{fin.dim}(A)$ the finitistic projective dimension of A , $\text{fin.inj.dim}(A)$ the finitistic injective dimension of A , and $\text{pd}_A(AeA)$ the projective dimension of AeA as a left A -module. Now our main result can be stated as follows.

Theorem *Let e be an idempotent in an Artin algebra A . Suppose that AeA is a strongly idempotent ideal with finite projective dimension. Then we have the following.*

- (1) $\text{fin.dim}(A) \leq \max\{2\text{fin.dim}(eAe) + 1, \text{pd}_A(AeA) + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2\}$;
- (2) $\text{fin.inj.dim}(A) \leq \text{pd}_A(AeA) + \text{fin.inj.dim}(eAe) + \text{fin.inj.dim}(A/AeA) + 2$.

In particular, if the finitistic projective (or injective) dimension of eAe and A are finite, then the finitistic projective (or injective) dimension of A is finite.

The theorem is a generalized version of [2, Theorem 5.4]. We will prove it in Theorems 3.5 and 3.11. Suppose that e is a primitive idempotent in A . Then eAe is a local algebra. Recall that the finitistic dimension of a local Artin algebra is zero. Immediately from the theorem, we get the following result as a corollary, which implies that both the finitistic projective and the finitistic injective dimension of standardly stratified algebras are finite (see [1]).

Corollary [1] *Let e be a primitive idempotent in A . Suppose that AeA is projective. Then we have the following.*

- (1) $\text{fin.dim}(A) \leq \text{fin.dim}(A/AeA) + 2$.
- (2) $\text{fin.inj.dim}(A) \leq \text{fin.inj.dim}(A/AeA) + 2$.

The paper is organized as follows. In Section 2, we give some notations, definitions and known results needed in the proof of the theorem. In Section 3, we prove basic results in the first subsection; then we give the proof of the main result in the next two subsections; finally, we prove general results in the last subsection.

2 preliminaries

In this section, we give some notations, definitions and known results needed in the proof of the theorem.

2.1 Notations and definitions

Let A be an Artin algebra. We denote A^{op} the opposite algebra of A . Unless otherwise specified, all modules considered are finitely generated left A -modules. Let X be an A -module and n be an integer with $n > 0$. The projective dimension of X is denoted by $\text{pd}_A(X)$; the n -th syzygy of X is denoted by $\Omega_A^n(X)$. The finitistic

projective dimension, or simply the finitistic dimension of A , which is denoted by $\text{fin.dim}(A)$, is defined as the supremum of the projective dimension of finitely generated A -modules with finite projective dimension. The finitistic injective dimension of A , which is denoted by $\text{fin.inj.dim}(A)$, is defined as the supremum of the injective dimension of finitely generated A -modules with finite injective dimension. An element e in A is called an idempotent if $e^2 = e$. An ideal I in A is called an **idempotent ideal** if there exists an idempotent e in A such that $I = AeA$. Now suppose that e is an idempotent in A and write $I = AeA$. Let X be an A/I -module. Then the epimorphism $A \rightarrow A/I$ induces a natural A -module structure of X . Let X and Y be two A/I -modules. Then there is an isomorphism $\text{Hom}_{A/I}(X, Y) \simeq \text{Hom}_A(X, Y)$. The isomorphism induces morphisms $\rho_{X,Y}^n : \text{Ext}_{A/I}^n(X, Y) \rightarrow \text{Ext}_A^n(X, Y)$ for all $n > 0$ and for all A/I -modules X and Y . The ideal I is called a **strongly idempotent ideal** if, for all A/I -modules X and Y , $\rho_{X,Y}^n$ are isomorphisms for all $n \geq 0$ (see [2]). It is proved in [4] that AeA is a strongly idempotent ideal if and only if

- (a) the multiplication map $Ae \otimes_{eAe} eA \rightarrow AeA$ is an isomorphism and
- (b) $\text{Tor}_n^{eAe}(Ae, eA) = 0$ for all $n > 0$.

The ideal satisfying the equivalent condition is also called a stratifying ideal in [4]. An Artin algebra A is called a **CPS-stratified algebra** if there exists a chain

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = A$$

of idempotent ideals in A such that I_{k+1}/I_k is a strongly idempotent ideal of A/I_k and that I_{k+1}/I_k is generated by a primitive idempotent in A/I_k for $0 \leq k \leq n-1$. In addition, if I_{k+1}/I_k is a projective A/I_k -module for $0 \leq k \leq n-1$, then A is called a **standardly stratified algebra**. For more details about the definition, we refer to [4].

In the rest of the paper, we will write $\bar{A} = A/AeA$ and $B = eAe$ for abbreviation sometimes.

2.2 Known results

In this subsection, we collect some results from [2], which will be needed in the proof of our main results.

Let e be an idempotent in A . The following lemma gives more characteristics of a strongly idempotent ideal.

Lemma 2.1 [2, Proposition 1.3] *The following statements are equivalent.*

- (1) *The ideal AeA is a strongly idempotent ideal.*
- (2) *The epimorphism $A \rightarrow A/AeA$ induces isomorphisms $\text{Tor}_n^A(X, Y) \rightarrow \text{Tor}_n^{\bar{A}}(X, Y)$ for all \bar{A} -modules X and Y , and for all $n > 0$.*
- (3) *$\text{Tor}_n^A(A/AeA, Y) = 0$ for all \bar{A} -modules Y and all $n > 0$.*

Denote $\text{add}(Ae)$ the full subcategory of the category of A -modules whose objects are direct summands of direct sums of finite copies of Ae . Let X be an A -module and

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow X \rightarrow 0$$

the minimal projective resolution of X . Let k be a naturel number. We say that X is in \mathbf{P}_e^k if P_i is in $\text{add}(Ae)$ for $0 \leq i \leq k$; and say that X is in \mathbf{P}_e^∞ if X is in \mathbf{P}_e^k for all $k \geq 0$. It is known from [2, Theorem 2.1] that AeA is a strongly idempotent ideal if and only if AeA is in \mathbf{P}_e^∞ . It follows that if AeA is projective on either side, then it is a strongly idempotent ideal. The following lemma provides us a criteria for determining whether an A -module is in \mathbf{P}_e^∞ .

Lemma 2.2 [2, Propsition 2.4] *The following conditions are equivalent for an A -module X .*

- (1) *The A -module X is in \mathbf{P}_e^∞ .*
- (2) *$\text{Tor}_n^A(A/AeA, X) = 0$ for all $n \geq 0$.*

Proof. For convenience, we include a proof here. It suffices to show that (2) implies (1). Let X be an A -module such that $\text{Tor}_n^A(A/AeA, X) = 0$ for all $n \geq 0$. Let

$$\cdots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

be the projective resolution of X . Since $A/AeA \otimes_A X = 0$, we get that P_0 is in $\text{add}(Ae)$. It follows that $A/AeA \otimes_A P_0 = 0$. Applying the functor $A/AeA \otimes_A -$ to the exact sequence $0 \rightarrow \Omega_A(X) \rightarrow P_0 \rightarrow X \rightarrow 0$, we get an exact sequence

$$0 \longrightarrow \text{Tor}_1^A(A/AeA, X) \longrightarrow A/AeA \otimes_A \Omega_A(X) \longrightarrow A/AeA \otimes_A P_0 \longrightarrow A/AeA \otimes_A X \longrightarrow 0.$$

It follows from $\text{Tor}_1^A(A/AeA, X) = 0$ that $A/AeA \otimes_A \Omega_A(X) = 0$. Then we get that P_1 is in $\text{add}(Ae)$. Now the result can be shown inductively. \square

Finally, we need the following lemma.

Lemma 2.3 [2, Corollary 3.2] *Let X be an A -module. If X is in \mathbf{P}_e^∞ , then $\text{pd}_A(X) = \text{pd}_B(eX)$.*

3 Proof of the theorem

Throughout this section, we denote A an Artin algebra and e an idempotent in A , and always assume that AeA is a strongly idempotent ideal.

3.1 Basic results

In this subsection, we prove basic results which will be used in the following subsections.

Lemma 3.1 *Let X be an A -module. If $\text{Tor}_k^A(A/AeA, X) = 0$ for $k \geq 1$, then AeX is in \mathbf{P}_e^∞ .*

Proof. Let X be an A -module such that $\text{Tor}_k^A(A/AeA, X) = 0$ for $k \geq 1$. Since AeA is a strongly idempotent ideal and X/AeX is an \overline{A} -module, we get from Lemma 2.1 that $\text{Tor}_k^A(A/AeA, X/AeX) = 0$ for $k \geq 1$. Applying the functor $\text{Hom}_A(A/AeA, -)$ to the exact sequence

$$0 \rightarrow AeX \rightarrow X \rightarrow X/AeX \rightarrow 0,$$

we get a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_{k+1}^A(A/AeA, X) \longrightarrow \operatorname{Tor}_{k+1}^A(A/AeA, X/AeX) \longrightarrow \operatorname{Tor}_k^A(A/AeA, AeX) \longrightarrow \operatorname{Tor}_k^A(A/AeA, X) \longrightarrow \cdots.$$

It follows that $\operatorname{Tor}_k^A(A/AeA, AeX) = 0$ for $k \geq 1$. Note that $(A/AeA) \otimes_A AeX = 0$. Consequently, $\operatorname{Tor}_k^A(A/AeA, AeX) = 0$ for $k \geq 0$. It follows from Lemma 2.2 that AeX is in \mathbf{P}_e^∞ . \square

Now we can prove the following result, which will be frequently used in the proof of the main result.

Lemma 3.2 *Let X be an A -module. Suppose that there exists a naturel number n such that $\operatorname{Tor}_k^B(Ae, eX) = 0$ for $k \geq n + 1$. Then $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ .*

Proof. Denote D the usual duality for Artin algebras. Let Z be an A^{op} -module and Y an A -module. It is known that there is an isomorphism $\operatorname{Tor}_k^A(Z, Y) \simeq D\operatorname{Ext}_{A^{op}}^k(Z, D(Y))$ for each $k \geq 0$. Let X be an A -module and m a naturel number. Then we have isomorphisms

$$\begin{aligned} \operatorname{Tor}_m^B(Ae, eX) &\simeq D\operatorname{Ext}_{B^{op}}^m(Ae, D(eX)) \\ &\simeq D\operatorname{Ext}_{B^{op}}^m(\operatorname{Hom}_{A^{op}}(eA, AeA), \operatorname{Hom}_{A^{op}}(eA, D(X))) \\ &\simeq D\operatorname{Ext}_{A^{op}}^m(AeA, D(X)) \\ &\simeq \operatorname{Tor}_m^A(AeA, X), \end{aligned}$$

where the third isomorphism follows from [2, Theorem 3.2] and the fact that AeA is in \mathbf{P}_e^∞ . By assumption, there exists a naturel number n such that $\operatorname{Tor}_k^B(Ae, eX) = 0$ for $k \geq n + 1$. Then $\operatorname{Tor}_k^A(AeA, X) = 0$ for $k \geq n + 1$. It follows that $\operatorname{Tor}_k^A(A/AeA, X) = 0$ for $k \geq n + 2$. Therefore, $\operatorname{Tor}_k^A(A/AeA, \Omega_A^{n+1}(X)) = 0$ for $k \geq 1$. It follows from Lemma 3.1 that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . \square

3.2 Finitistic projective dimensions

In this subsection, we always assume that AeA is a strongly idempotent ideal with finite projective dimension. This subsection aims to prove that the finiteness of the finitistic dimension of A/AeA and eAe implies that of the finitistic dimension of A under this condition.

We need the following homological fact first. For completeness, we include a proof here.

Lemma 3.3 *Let X be an A -module. If $\operatorname{pd}_A(X) < +\infty$, then $\operatorname{pd}_B(eX) < +\infty$*

Proof. Let X be an A -module with finite projective dimension. Let

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

be the projective resolution of X . Applying the functor $\operatorname{Hom}(Ae, -)$ to it, we get a long exact sequence

$$0 \longrightarrow eP_m \longrightarrow eP_{m-1} \longrightarrow \cdots \longrightarrow eP_0 \longrightarrow eX \longrightarrow 0$$

of B -modules. Since the strongly idempotent ideal AeA is in \mathbf{P}_e^∞ , we know from Lemma 2.3 that $\operatorname{pd}_B(eA) = \operatorname{pd}_A(AeA)$. Since $\operatorname{pd}_A(AeA) < +\infty$, we have $\operatorname{pd}_B(eA) < +\infty$. It follows from the previous long exact sequence that $\operatorname{pd}_B(eX) < +\infty$. \square

Remark 3.1 If we drop the condition that AeA has finite projective dimension, then the lemma does not have to be true. For example, let A be the algebra given by the following quiver with relation.

$$\begin{array}{ccc} 1 & \xrightleftharpoons[\alpha]{\beta} & 2 \\ & & \alpha\beta\alpha=0. \end{array}$$

Then the indecomposable projective modules of A are as follows:

$$\begin{array}{cc} 1 & 2 \\ 2 & 1 \\ 1 & 2 \\ 2 & \end{array}$$

Let S be the simple A -module corresponding to the vertex 1 and e_1 the idempotent corresponding to the vertex 1. It is easy to check that Ae_1A is in \mathbf{P}_e^∞ . It follows that Ae_1A is a strongly idempotent ideal with infinite projective dimension. One can check that $\text{pd}_A(S) = 1$ but $\text{pd}_B(e_1S)$ is infinite.

Lemma 3.4 *Let X be an A -module with finite projective dimension. Then there exists a natural number n with $n \leq \text{fin.dim}(B)$ such that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ .*

Proof. Let X be an A -module with finite projective dimension. Then we get from Lemma 3.3 that the projective dimension of eX is finite. Suppose that $\text{pd}_B(eX) = n$. Then $\text{Tor}_k^B(Ae, eX) = 0$ for $k \geq n+1$. Since AeA is a strongly idempotent ideal, we get from Lemma 3.2 that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . \square

Now we can prove the first main result in the paper. Although we can also prove the next theorem by Lemma 3.7 in the next subsection, we include a different proof here, not only since the upper bound here is better but also because the proof may be of its own interest.

Theorem 3.5 *Let e be an idempotent in A such that AeA is a strongly idempotent ideal with finite projective dimension. Then we have the following.*

- (1) $\text{fin.dim}(A) \leq \max\{2\text{fin.dim}(eAe) + 1, \text{pd}_A(AeA) + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2\}$
- (2) $\text{fin.dim}(A) \leq 2\text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2.$

In particular, if $\text{fin.dim}(A/AeA) < +\infty$ and $\text{fin.dim}(eAe) < +\infty$, then $\text{fin.dim}(A) < +\infty$.

Proof. (1) Let X be an A -module with finite projective dimension. We know from Lemma 3.4 that there exists a natural number n with $n \leq \text{fin.dim}(B)$ such that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . It follows from Lemma 2.3 that $\text{pd}_A(Ae\Omega_A^{n+1}(X)) = \text{pd}_B(e\Omega_A^{n+1}(X))$. Since $\text{pd}_A(\Omega_A^{n+1}(X)) < +\infty$, we get from Lemma 3.3 that $\text{pd}_B(e\Omega_A^{n+1}(X)) < +\infty$. It follows that $\text{pd}_A(Ae\Omega_A^{n+1}(X)) < +\infty$. As a result, $\text{pd}_A(\Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X)) < +\infty$. Since AeA is a strongly idempotent ideal, we know that $\text{pd}_{\bar{A}}(Y) \leq \text{pd}_A(Y)$ for any \bar{A} -module Y . It follows that $\text{pd}_{\bar{A}}(\Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X))$ is finite. On the other hand, we get from the change of rings theorem that $\text{pd}_A(Y) \leq \text{pd}_{\bar{A}}(Y) + \text{pd}_A(\bar{A})$ for any \bar{A} -module Y . Then we have

$$\begin{aligned} \text{pd}_A(\Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X)) &\leq \text{pd}_{\bar{A}}(\Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X)) + \text{pd}_A(A/AeA) \\ &\leq \text{pd}_{\bar{A}}(\Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X)) + \text{pd}_A(AeA) + 1 \end{aligned}$$

Considering the exact sequence $0 \rightarrow Ae\Omega_A^{n+1}(X) \rightarrow \Omega_A^{n+1}(X) \rightarrow \Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X) \rightarrow 0$, we get that

$$\begin{aligned}
\text{pd}_A(X) &\leq n+1 + \text{pd}_A(\Omega_A^{n+1}(X)) \\
&\leq n+1 + \max\{\text{pd}_A(Ae\Omega_A^{n+1}(X)), \text{pd}_A(\Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X))\} \\
&\leq n+1 + \max\{\text{pd}_A(Ae\Omega_A^{n+1}(X)), \text{pd}_A(\Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X)) + \text{pd}_A(AeA) + 1\} \\
&= n+1 + \max\{\text{pd}_B(e\Omega_A^{n+1}(X)), \text{pd}_A(\Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X)) + \text{pd}_A(AeA) + 1\} \\
&\leq \text{fin.dim}(eAe) + 1 + \max\{\text{fin.dim}(eAe), \text{fin.dim}(A/AeA) + \text{pd}_A(AeA) + 1\} \\
&= \max\{2\text{fin.dim}(eAe) + 1, \text{pd}_A(AeA) + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2\}
\end{aligned}$$

where the equality follows from Lemma 2.3 and the fact that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . Consequently,

$$\text{fin.dim}(A) \leq \max\{2\text{fin.dim}(eAe) + 1, \text{pd}_A(AeA) + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2\}.$$

(2) Since the strongly idempotent ideal AeA is in \mathbf{P}_e^∞ . We get from Lemma 2.3 that $\text{pd}_A(AeA) = \text{pd}_{eAe}(eA)$. Then the result follows from (1). \square

Remark 3.2 (1) Let $0 = I_0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = A$ be a chain of idempotent ideals in A such that I_{k+1}/I_k is a projective A/I_k -module for $0 \leq k \leq n-1$. It is proved in [2, Proposition 6.1] that I_k is a strongly idempotent ideal with finite projective dimension for $1 \leq k \leq n$. This can provide us examples of strongly idempotent ideals which are not projective but of finite projective dimension.

(2) Let d be the supremum of the set $\{\text{pd}_A(X) \mid eX = 0\}$. In [6], it is proved that $\text{fin.dim}(A) \leq \text{fin.dim}(eAe) + d + 1$ under the condition that $\text{pd}_{eAe}eA < +\infty$. This is different from our result since the supremum does not have to be finite under the condition in the theorem.

Recall that if AeA is projective, then it is a strongly idempotent ideal. Combining with Theorem 3.5, we get the following result, which implies that the finitistic dimension of standardly stratified algebras is finite.

Corollary 3.6 [1, Theorem 2.2] *Let e be a primitive idempotent in A . Suppose that AeA is projective. Then $\text{fin.dim}(A) \leq \text{fin.dim}(A/AeA) + 2$.*

Proof. Since e is a primitive idempotent in A , we know that eAe is a local algebra. Then the result follows from Theorem 3.5 and the fact that the finitistic dimension of a local Artin algebra is zero. \square

3.3 Finitistic injective dimensions

This subsection is devoted to showing that the finiteness of the finitistic injective dimension of A/AeA and eAe implies that of the finitistic injective dimension of A under the condition that AeA has finite projective dimension.

We prove the following lemma first.

Lemma 3.7 *Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be an exact sequence of A -modules with Z an A/AeA -module. Suppose that X is in \mathbf{P}_e^∞ and that the projective dimension of Y is finite. Then we have the following.*

$$(1) \text{pd}_{\bar{A}}(Z) < +\infty.$$

$$(2) \text{pd}_A(Y) \leq \text{fin.dim}(A/AeA) + \text{fin.dim}(eAe) + 1.$$

Proof. (1) Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be an exact sequence of A -modules. Suppose that $\text{pd}_A(Y) = n < +\infty$. Then we get an exact sequence $0 \rightarrow \Omega_A^n(X) \rightarrow \Omega_A^n(Y) \oplus Q \rightarrow \Omega_A^n(Z) \rightarrow 0$ with $\Omega_A^n(Y) \oplus Q$ an projective A -module. It follows that $\Omega_A^{n+2}(Z) \simeq \Omega_A^{n+1}(X)$. Since X is in \mathbf{P}_e^∞ , we get that $\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . It follows that $\Omega_A^{n+2}(Z)$ is in \mathbf{P}_e^∞ . Suppose that Z is an A/AeA -module. Let

$$\cdots \rightarrow P_{n+2} \rightarrow P_{n+1} \rightarrow \cdots \rightarrow P_0 \rightarrow Z \rightarrow 0$$

be the minimal projective resolution of Z as an A -module. Since AeA is a strongly idempotent ideal, we get from Lemma 2.1 that $\text{Tor}_k^A(A/AeA, Z) = 0$ for $k \geq 0$. Applying the functor $A/AeA \otimes_A -$ to the resolution, we get a long exact sequence

$$\cdots \rightarrow A/AeA \otimes_A P_{n+2} \rightarrow A/AeA \otimes_A P_{n+1} \rightarrow \cdots \rightarrow A/AeA \otimes_A P_0 \rightarrow Z \rightarrow 0,$$

which is the minimal projective resolution of Z as an A/AeA -module since the functor $A/AeA \otimes_A -$ preserves projective covers. We have seen that $\Omega_A^{n+2}(Z)$ is in \mathbf{P}_e^∞ . It follows that $A/AeA \otimes_A P_{n+2} = 0$. Therefore, $\text{pd}_{\bar{A}}(Z) \leq n + 1 < +\infty$.

(2) It follows from (1) that $\text{pd}_{\bar{A}}(Z) < +\infty$. Suppose $\text{pd}_{\bar{A}}(Z) = m$. Comparing the long exact sequences in the proof of (1), we know that $A/AeA \otimes_A P_k = 0$ for $k \geq m + 1$. It follows that $\Omega_A^{m+1}(Z)$ is in \mathbf{P}_e^∞ . By assumption, X is in \mathbf{P}_e^∞ . Considering the exact sequence $0 \rightarrow \Omega_A^{m+1}(X) \rightarrow \Omega_A^{m+1}(Y) \oplus R \rightarrow \Omega_A^{m+1}(Z) \rightarrow 0$ with R a projective A -module, we get that $\Omega_A^{m+1}(Y)$ is in \mathbf{P}_e^∞ since \mathbf{P}_e^∞ is closed under extensions. Then we have by Lemma 2.3 that $\text{pd}_A(\Omega_A^{m+1}(Y)) = \text{pd}_B(e\Omega_A^{m+1}(Y))$. It follows that

$$\text{pd}_A(Y) \leq m + 1 + \text{pd}_A(\Omega_A^{m+1}(Y)) = m + 1 + \text{pd}_B(e\Omega_A^{m+1}(Y)) \leq \text{fin.dim}(A/AeA) + \text{fin.dim}(eAe) + 1.$$

This finishes the proof of the lemma. \square

Immediately from the lemma, we get the following result.

Proposition 3.8 *Suppose that AeA is a strongly idempotent ideal. Denote $\text{fin.dim}_A(A/AeA)$ the supremum of the set $\{\text{pd}_A(X) \mid eX = 0 \text{ and } \text{pd}_A(X) < +\infty\}$. Then $\text{fin.dim}_A(A/AeA) \leq \text{fin.dim}(A/AeA) + \text{fin.dim}(eAe) + 1$.*

Proof. Let X be an \bar{A} -module. Then $AeX = 0$. It follows from Lemma 3.7 that

$$\text{pd}_A(X) \leq \text{fin.dim}(A/AeA) + \text{fin.dim}(eAe) + 1.$$

This finishes the proof. \square

Lemma 3.9 *Suppose that AeA is a strongly idempotent ideal with finite projective dimension as a right A -module. Suppose that $\text{pd}_{A^{op}}(AeA) = n$. Let X be an A -module. Then $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ .*

Proof. Let X be an A -module. Suppose that $\text{pd}_{A^{op}}(AeA) = n$, we get from Lemma 2.3 that $\text{pd}_{B^{op}}(Ae) = n$. Thus $\text{Tor}_k^B(Ae, eX) = 0$ for $k \geq n+1$. It follows from Lemma 3.2 that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . \square

Now we can prove the following result, which is a generalized version of [2, Theorem 5.4].

Proposition 3.10 *Suppose that AeA is a strongly idempotent ideal with finite projective dimension as a right A -module. Then*

$$\text{fin.dim}(A) \leq \text{pd}_{A^{op}}(AeA) + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2.$$

In particular, if $\text{fin.dim}(A/AeA) < +\infty$ and $\text{fin.dim}(eAe) < +\infty$, then $\text{fin.dim}(A) < +\infty$.

Proof. Let X be an A -module with finite projective dimension. Suppose that $n = \text{pd}_{A^{op}}(AeA)$. It follows from Lemma 3.9 that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . Then the exact sequence

$$0 \longrightarrow Ae\Omega_A^{n+1}(X) \longrightarrow \Omega_A^{n+1}(X) \longrightarrow \Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X) \longrightarrow 0$$

satisfies all the conditions in Lemma 3.7. It follows that $\text{pd}_A(\Omega_A^{n+1}(X)) \leq \text{fin.dim}(A/AeA) + \text{fin.dim}(eAe) + 1$. Then we have

$$\text{pd}_A(X) \leq n+1 + \text{pd}_A(\Omega_A^{n+1}(X)) \leq n + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2.$$

Consequently, $\text{fin.dim}(A) \leq \text{pd}_{A^{op}}(AeA) + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2$. \square

Remark 3.3 Let A be the algebra given in Remark 3.1. Let e_2 be the idempotent corresponding to the vertex 2. Then we see that Ae_2A is projective as a left A -module. But the projective dimension of Ae_2A as a right A -module is infinite. This shows that the previous proposition can not be deduced from Theorem 3.5.

Now we can prove the following.

Theorem 3.11 *Suppose that AeA is a strongly idempotent ideal with finite projective dimension. Then*

$$\text{fin.inj.dim}(A) \leq \text{pd}_A(AeA) + \text{fin.inj.dim}(eAe) + \text{fin.inj.dim}(A/AeA) + 2.$$

In particular, if $\text{fin.inj.dim}(A/AeA) < +\infty$ and $\text{fin.inj.dim}(eAe) < +\infty$, then $\text{fin.inj.dim}(A) < +\infty$.

Proof. Suppose that AeA is a strongly idempotent ideal with finite projective dimension. Then $\text{pd}_{(A^{op})^{op}}(AeA)$ is finite. It follows from Proposition 3.10 that

$$\text{fin.dim}(A^{op}) \leq \text{pd}_A(AeA) + \text{fin.dim}((eAe)^{op}) + \text{fin.dim}((A/AeA)^{op}) + 2.$$

Then the result follows from the fact that $\text{fin.dim}(\Lambda^{op}) = \text{fin.inj.dim}(\Lambda)$ for any Artin algebra Λ . \square

Immediately from the previous theorem, similarly as in Corollary 3.6, we get the following result, which implies that the finitistic injective dimension of standardly stratified algebras is finite (see [1]).

Corollary 3.12 *Let e be a primitive idempotent in A . Suppose that AeA is projective. Then*

$$\text{fin.inj.dim}(A) \leq \text{fin.inj.dim}(A/AeA) + 2.$$

3.4 General cases

In this subsection, we drop the condition that AeA has finite projective dimension and strengthen other conditions. We prove the following general result first.

Proposition 3.13 *Suppose that AeA is a strongly idempotent ideal and that there exists a naturel number n such that $\text{Tor}_k^B(Ae, eX) = 0$ for all A -modules X with finite projective dimension and all $k \geq n+1$. Then*

$$\text{fin.dim}(A) \leq n + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2.$$

In particular, if $\text{fin.dim}(A/AeA) < +\infty$ and $\text{fin.dim}(eAe) < +\infty$, then $\text{fin.dim}(A) < +\infty$.

Proof. Let X be an A -module with finite projective dimension. By assumption, $\text{Tor}_k^B(Ae, eX) = 0$ for $k \geq n+1$. It follows from Lemma 3.2 that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . Considering the exact sequence

$$0 \longrightarrow Ae\Omega_A^{n+1}(X) \longrightarrow \Omega_A^{n+1}(X) \longrightarrow \Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X) \longrightarrow 0,$$

we get from Lemma 3.7 that

$$\text{pd}_A(X) \leq n+1 + \text{pd}_A(\Omega_A^{n+1}(X)) \leq n + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2.$$

Thus, $\text{fin.dim}(A) \leq n + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2$. Consequently, if $\text{fin.dim}(A/AeA) < +\infty$ and $\text{fin.dim}(eAe) < +\infty$, then $\text{fin.dim}(A) < +\infty$. \square

Let A be an Artin algebra and n an integer with $n \geq 1$. Set

$$\Omega_A^n(A\text{-mod}) := \{\Omega_A^n(X) \mid X \in A\text{-mod}\}.$$

Then we say that A is ***-syzygy finite** if there exists a naturel number m such that there exist only finitely many non-isomorphic indecomposable modules in $\Omega_A^m(A\text{-mod})$. Recall that an Artin algebra A is said to be of **finite representation type** if there exist only finitely many non-isomorphic indecomposable A -modules. Thus, an algebra which is representation type is *-syzygy finite. Obviously, if A is *-syzygy finite, then $\text{fin.dim}(A) < +\infty$.

Proposition 3.14 *Suppose that AeA is a strongly idempotent ideal such that eAe is *-syzygy finite. If $\text{fin.dim}(A/AeA) < +\infty$, then $\text{fin.dim}(A) < +\infty$.*

Proof. Suppose that eAe is *-syzygy finite. Then there exists a naturel number n such that there are only finitely many non-isomorphic indecomposable modules in $\Omega_B^n(B\text{-mod})$. Set

$$\mathcal{M} = \{Y \in \Omega_B^n(B\text{-mod}) \mid \text{there exists a naturel number } l \text{ such that } \text{Tor}_k^B(Ae, Y) = 0 \text{ for } k \geq l+1\}.$$

Then there exists a naturel number p such that $\text{Tor}_k^B(Ae, Y) = 0$ for all $Y \in \mathcal{M}$ and for all $k \geq p+1$. Let X be an A -module with $\text{pd}_A(X) = m \geq n+2$. Let

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

be the minimal projective resolution of X . Applying the functor $\text{Hom}(Ae, -)$ to it, we get a long exact sequence

$$0 \longrightarrow eP_m \longrightarrow eP_{m-1} \longrightarrow \cdots \longrightarrow eP_0 \longrightarrow eX \longrightarrow 0$$

of B -modules. Since AeA is a strongly idempotent ideal, we get that $\text{Tor}_k^B(Ae, eA) = 0$ for $k \geq 1$. Thus, $\text{Tor}_k^B(Ae, eP_m) = \text{Tor}_k^B(Ae, eP_{m-1}) = 0$ for $k \geq 1$. It follows that $\text{Tor}_k^B(Ae, e\Omega_A^{m-1}(X)) = 0$ for $k \geq 2$. Inductively, one can prove that $\text{Tor}_k^B(Ae, eX) = 0$ for $k \geq m+1$. Then we have $\text{Tor}_k^B(Ae, \Omega_B^n(eX)) = 0$ for $k \geq m-n+1$. It follows that $\Omega_B^n(eX) \in \mathcal{M}$. Consequently, $\text{Tor}_k^B(Ae, \Omega_B^n(eX)) = 0$ for $k \geq p+1$. As a result, $\text{Tor}_k^B(Ae, eX) = 0$ for $k \geq n+p+1$. Note that $\text{fin.dim}(eAe) < +\infty$ since it is $*$ -syzygy finite. It follows from Proposition 3.13 that if $\text{fin.dim}(A/AeA) < +\infty$, then $\text{fin.dim}(A) < +\infty$. \square

As a direct consequence of the proposition, we get the following.

Corollary 3.15 *Suppose that AeA is a strongly idempotent ideal and that the finitistic dimension of A/AeA is finite. Then the finitistic dimension of A is finite if one of the following conditions holds.*

- (1) *The global dimension of eAe is finite.*
- (2) *The algebra eAe is of finite representation type.*
- (3) *The algebra eAe is monomial.*

Proof. By Proposition 3.14, it suffices to show that eAe is $*$ -syzygy finite under each condition. This is clear for (1) and (2) and it follows from [12, Theorem I] that a monomial algebra is $*$ -syzygy finite. This completes the proof. \square

Recall that an A -module X is said to be **Gorenstein projective** if there exists an exact sequence

$$P^\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective modules with $X \simeq \text{Im}(P_0 \longrightarrow P^0)$ such that $\text{Hom}_A(P^\bullet, A)$ is exact for all projective modules Q . Let n be a naturel number. We say that the Gorenstein projective dimension of X is at most n if X has a Gorenstein projective resolution of length n . For more details on the definition, we refer to [7].

Proposition 3.16 *Suppose that AeA is a strongly idempotent ideal such that the Gorenstein projective dimension of AeA_A is finite. If $\text{fin.dim}(A/AeA) < +\infty$ and $\text{fin.dim}(eAe) < +\infty$, then $\text{fin.dim}(A) < +\infty$.*

Proof. Suppose that the Gorenstein projective dimension of AeA_A is at most n . Let X be an A -module with finite projective dimension. Then by [8, Lemma 4.1], we know that $\text{Tor}_k^A(AeA, X) = 0$ for $k \geq n+1$. It follows that $\text{Tor}_k^A(A/AeA, X) = 0$ for $k \geq n+2$, and therefore $\text{Tor}_k^A(A/AeA, \Omega_A^{n+1}(X)) = 0$ for $k \geq 1$. It follows from Lemma 3.1 that $Ae\Omega_A^{n+1}(X)$ is in \mathbf{P}_e^∞ . Considering the exact sequence

$$0 \longrightarrow Ae\Omega_A^{n+1}(X) \longrightarrow \Omega_A^{n+1}(X) \longrightarrow \Omega_A^{n+1}(X)/Ae\Omega_A^{n+1}(X) \longrightarrow 0,$$

we get from Lemma 3.7 that

$$\text{pd}_A(X) \leq n+1 + \text{pd}_A(\Omega_A^{n+1}(X)) \leq n + \text{fin.dim}(eAe) + \text{fin.dim}(A/AeA) + 2.$$

Then the result follows from that assumption that the finitistic dimension of eAe and A/AeA are finite. \square

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